References

original papers for Coulomb branches

Braverman-Finkelberg-N, 1601.03586, 1604.03625, 1706.05154

expository papers

N, 1706.05154 BF, 1807.09038 F, 1712.03039

Bow varieties and Coulomb branch

N-Takayama, 1606.02002

geometric Satake for affine Lie algebras of type A

N, 1810.04293

References in physics literature are omitted. See the list in 1503.03676.

§. prelude

convolution

$$\begin{array}{ll} M & : \text{smooth} \\ \pi \downarrow & \text{proper} \\ X & \end{array} \qquad Z = M \times_X M$$

$H_*(Z)$ is equipped with an algebra structure :

$$c, c' \in H_*(Z) \qquad c \cdot c' = p_{13*}(\, p_{12}^* c \cap p_{23}^* c')$$

If $G \curvearrowright (M \xrightarrow{\pi} X)$, we can also consider $H_*^G(Z)$.

This construction has been used to realize various algebras :

- degenerate affine Hecke algebra (Lusztig) Weyl group Springer ...
- Yangian (Varagnolo) Kac-Moody N .... Ginzburg ..

slightly generalize : $\underset{\text{reductive}}{G} \supset \underset{\text{parabolic}}{P}$ $V \supset V'$

representation of G $\qquad$ P-invariant subspace

$$M = G \times^P V' \xrightarrow{\pi} X = V \qquad Z = \{([g_1, v_1], [g_2, v_2]) \mid g_1 v_1 = g_2 v_2\}$$

$$[g, v] \longmapsto gv$$

mathematical definition of Coulomb branches [Braverman – Finkelberg – N]

- Construct commutative algebras by convolution
  (Then define affine algebraic varieties as their Spec)

But we usually get noncommutative algebras.

We are interested in representation theory of convolution algebras

How to find commutative algebras by convolution?

- Use affine Grassmannian $Gr_G = G(K) / G(\mathcal{O})$ $K = \mathbb{C}((z)) \supset \mathcal{O} = \mathbb{C}[[z]]$
  (Xinwen's lectures)

  $G(K)$: analog of G $G(\mathcal{O})$: analog of P

heuristic explanation • $W_{aff} = W \ltimes \underbrace{Q}_{\text{commutative}}$

- geometric Satake realises $(\mathrm{Rep}\, G^\vee, \underbrace{\otimes}_{\text{commutative}})$ by convolution

☆ Bonus Get noncommutative deformation by loop rotation on Gr_G.

§1. Construction

G : complex reductive group

M : symplectic representation of G $\qquad$ ω : sympl. form is preserved by G

Assume $\quad M = N \oplus N^*$ $\qquad$ N : cpx representation of G

in the whole lectures.

general M $\quad$ Braverman–Dhillon–Finkelberg–Raskin–Travkin 2201.09475

Teleman 2209.01088

$$\mathcal{K} = \mathbb{C}((z)) \supset \mathcal{O} = \mathbb{C}[[z]]$$

$$\begin{tikzcd} N(\mathcal{K}) & G(\mathcal{K}) \arrow[l, bend right] \\ N(\mathcal{O}) \arrow[u, phantom, "\cup"] & G(\mathcal{O}) \arrow[u, phantom, "\cup"] \arrow[l, bend left, "\text{preserved}"'] \end{tikzcd}$$

$$\mathcal{T} \overset{\text{def.}}{=} G(\mathcal{K}) \times^{G(\mathcal{O})} N(\mathcal{O}) \longrightarrow N(\mathcal{K})$$

$$[g(z), s(z)] \longmapsto g(z)s(z)$$

$G(\mathcal{K}) \times^{G(\mathcal{O})} N(\mathcal{O})$: analog of M $\qquad$ $N(\mathcal{K})$: analog of X

Then we consider $\quad H_*^{G(\mathcal{K})}(\mathcal{T} \times_{N(\mathcal{K})} \mathcal{T})$

$$\mathcal{T} \times_{N(\mathcal{K})} \mathcal{T} = \{([g_1(z), s_1(z)], [g_2(z), s_2(z)]) \mid g_1(z)s_1(z) = g_2(z)s_2(z)\}$$

However, we cannot justify the definition of $H_*^{G(K)}(\mathcal{J} \times_{N(K)} \mathcal{J})$.

Observe $[G(K) \backslash \mathcal{J} \times_{N(K)} \mathcal{J}] = [G(\mathcal{O}) \backslash \mathcal{R}]$

where $\mathcal{R} = \{([g_1(z), s_1(z)], [id, g(z)s_1(z)]) \mid g_1(z)s_1(z) \in N(\mathcal{O})\}$

$\therefore$ We consider $H_*^{G(\mathcal{O})}(\mathcal{R})$ instead.

<u>Th</u> 1 (1) $H_*^{G(\mathcal{O})}(\mathcal{R})$ is well-defined such that [fiber over $1 \in Gr_G$] has degree $= 0$

(2) $H_*^{G(\mathcal{O})}(\mathcal{R})$ is equipped with a convolution product s.t. is the unit.

It is $H^*_{G(\mathcal{O})}(pt) = H^*_G(pt)$ - linear in the 1st variable.

$(fc_1) \cdot c_2 = fc_1 \cdot c_2 \qquad f \in H^*_G(pt)$

(Since $\bullet$ is commutative, it is also so in the 2nd variable.
But it is no longer true for quantization.)

(Sketch of proof) Modify MV diagram:

$$
\begin{tikzcd}
Gr_G \times Gr_G & G(K) \times Gr_G \arrow[l] \arrow[r] & G(K) \times^{G(\mathcal{O})} Gr_G \arrow[r] & Gr_G \\
\mathcal{T} \times \mathcal{R} \arrow[u] & G(K) \times \mathcal{R} \arrow[l, "p"'] \arrow[r, "q"] \arrow[u] & G(K) \times^{G(\mathcal{O})} \mathcal{R} \arrow[r, "m"] \arrow[u] & \mathcal{T} \arrow[u] \\
([g_1, g_2 s], [g_2, s]) & (g_1, [g_2, s]) \arrow[l, mapsto] \arrow[r, mapsto] & [g_1, [g_2, s]] \arrow[r, mapsto] & [g_1 g_2, s] \\
\mathcal{R} \times \mathcal{R} \arrow[u, phantom, "\cup"] & p^{-1}(\mathcal{R} \times \mathcal{R}) \arrow[l] \arrow[r] \arrow[u, phantom, "\cup"] & q(p^{-1}(\mathcal{R} \times \mathcal{R})) \arrow[r] \arrow[u, phantom, "\cup"] & \mathcal{R} \arrow[u, phantom, "\cup"]
\end{tikzcd}
$$

$g_1 = g_1(z)$ etc

- : $H_*^{G(\mathcal{O})}(\mathcal{R}) \otimes H_*^{G(\mathcal{O})}(\mathcal{R}) \xrightarrow{p^*} H_*^{G(\mathcal{O}) \times G(\mathcal{O})}(p^{-1}(\mathcal{R} \times \mathcal{R})) \xrightarrow{(q^*)^{-1}} H_*^{G(\mathcal{O})}(q(p^{-1}(\mathcal{R} \times \mathcal{R}))) \xrightarrow{m_*} H_*^{G(\mathcal{O})}(\mathcal{R})$

⋆ check is well-defined.

⋆ check associativity ....

//

Remark Homological degree $\in \mathbb{Z}$, not necessarily $\mathbb{Z}_{\geq 0}$

quantization : $\mathbb{C}^\times \curvearrowright \mathcal{O}, K \qquad z \mapsto tz$ (loop rotation)

$G(K), G(\mathcal{O})$ etc

$\therefore H_*^{G(\mathcal{O}) \rtimes \mathbb{C}^\times}(\mathcal{R})$ + convolution product is defined.

$\uparrow$

$H^*_{\mathbb{C}^\times}(pt) = \mathbb{C}[\hbar]$ - algebra

$$H_*^{G(\mathcal{O}) \rtimes \mathbb{C}^\times}(\mathcal{R})\Big|_{\hbar=0} \cong H_*^{G(\mathcal{O})}(\mathcal{R})$$

Th 2. $\ast$ is commutative, but quantization is not.

There are two proofs (a) reduction to torus case then compute $\ast$ explicitly

(b) [Lonergan in Appendix of 3rd BFN] Use Beilinson-Drinfeld grassmannian

Def. (1) $\mathcal{M}_C \equiv \mathcal{M}_C(G, N) = \operatorname{Spec} H_*^{G(\mathcal{O})}(\mathcal{R})$ (Coulomb branch)

(2) Poisson bracket $\{c_1, c_2\} = \dfrac{[\tilde{c}_1, \tilde{c}_2]}{\hbar}\Big|_{\hbar=0}$

Comment : $\mathcal{M}_C$ is a priori an affine scheme.

But it can be shown $\mathcal{M}_C$: irreducible algebraic variety
normal.
important later.

integrable system

$H^*_{G(\mathcal{O})\rtimes\mathbb{C}^\times}(pt) \longrightarrow H_*^{G(\mathcal{O})\rtimes\mathbb{C}^*}(\mathcal{R})$ algebra homomorphism

commutative

$\rightsquigarrow \quad \mathcal{M}_C \xrightarrow{\varpi} \operatorname{Spec} H^*_G(pt) = \mathfrak{t}/W$ $\qquad$ $\mathfrak{t} = \operatorname{Lie} T \qquad T \subset G$

s.t. $\{\varpi^* f, \varpi^* g\} = 0$ $\qquad$ $W =$ Weyl grp $\qquad$ max torus

We will see that generic fiber of $\varpi = T^\vee$ (dual torus of T)

§2. Case $G = T$ torus

Example 1.

$G = \mathbb{C}^\times$, $N = 0$

$\mathcal{R} = \mathrm{Gr}_G = \mathrm{Gr}_{\mathbb{C}^\times} = \{ z^n \mid n \in \mathbb{Z} \}$ (reduced scheme structure)

$\therefore H_*^{G(\mathcal{O})}(\mathcal{R}) = \bigoplus_{n \in \mathbb{Z}} H^*_{\mathbb{C}^\times}(pt)[z^n] = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}[w] r_n$

convolution is given by $z^m \times z^n \longrightarrow z^{m+n}$

$\therefore \quad r_m \cdot r_n = r_{m+n}$

- is linear also in the 2^{nd} variable

$\therefore H_*^{G(\mathcal{O})}(\mathcal{R}) = \mathbb{C}[w, r_{\pm 1}]$ $\qquad \therefore \mathcal{M}_C = \mathrm{Spec} = \overset{w}{\mathbb{C}} \times \overset{r_1}{\mathbb{C}^\times}$

quantization $\quad r_n w - w r_n = ?$

$w = c_1(L_w)$
$\hbar = c_1(L_\hbar)$

part of diagram : $\underset{z^n}{G(K)} \times \mathrm{Gr}_G \xrightarrow{q} G(K) \times^{G(\mathcal{O})} \mathrm{Gr}_G$

L_w is twisted by $L_\hbar^{\otimes n}$ $\qquad \therefore ? = n \hbar r_n$

$$[r_n, w] = n\hbar r_n$$

$\therefore$ r_n is a difference operator : $w \longmapsto w + n\hbar$

<u>Remark</u> Two natural views of quantization :

1) difference operators on $\mathbb{C} \equiv \mathbb{C}_w$
2) differential operators on $\mathbb{C}^\times = \mathbb{C}^\times_{r_1}$ $\qquad w = -r_1 \dfrac{\partial}{\partial r_1}$

In view of integrable system $\mathbb{C}_w \times \mathbb{C}^\times_{r_1} \longrightarrow \mathbb{C}_w$,

1) is natural.

More intrinsically :

$$G = T, N = 0 \Rightarrow Gr_G \cong \mathrm{Hom}(\mathbb{C}^\times, T) = \mathrm{Hom}(T^\vee, \mathbb{C}^\times)$$

$$\therefore \mathcal{M}_C \cong \mathfrak{t} \times T^\vee = \text{cotangent bundle of } T^\vee$$

__Example 2.__ $G = \mathbb{C}^\times$, $N = \mathbb{C}$ (natural representation)

Let us first prepare :

Recall
$$\begin{tikzcd} \mathrm{Gr}_G \arrow[r, leftarrow] \arrow[r, bend right, "z"'] & \mathcal{T} & \mathcal{R} \arrow[l, hook', "i"'] \end{tikzcd}$$

zero section image is contained in $\mathcal{R}$

$$H_*^{G(\mathcal{O})}(\mathcal{R}) \xrightarrow{i_*} H_*^{G(\mathcal{O})}(\mathcal{T}) \underset{\cong}{\xrightarrow{z^*}} H_*^{G(\mathcal{O})}(\mathrm{Gr}_G)$$

(Thom isom.) $\quad$ $\mathcal{R}$ for Example 1

__Prop.__ The composite is an algebra homomorphism.

$r'_n :=$ fundamental class of the fiber at $\mathcal{R} \longrightarrow \mathrm{Gr}_G \ni z^n$

$$\cong$$

$$z^n \mathbb{C}[[z]] \cap \mathbb{C}[[z]] = z^{\max(0,n)} \mathbb{C}[[z]]$$

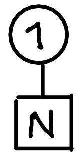
On the other hand, fiber of $\mathcal{T} \longrightarrow \mathrm{Gr}_G$ is $z^n \mathbb{C}[[z]]$

$$\therefore \ r'_n = \begin{cases} w^{-n} r_n & n < 0 \\ r_n & n \geq 0 \end{cases} \qquad \underset{x}{\underset{\|}{r'_1}} \cdot \underset{y}{\underset{\|}{r'_{-1}}} = w\, r_1 r_{-1} = w$$

$$\therefore \ \mathcal{M}_C = \mathrm{Spec} = \mathbb{C}^2$$

Example 3

$G = \mathbb{C}^\times$, $N = \mathbb{C}^N = (\mathrm{wt}\ 1)^{\oplus N}$

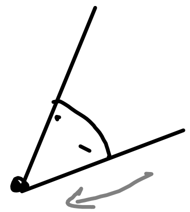


Same calculation : $r_n' = \begin{cases} w^{Nn} r_n & \text{if } n < 0 \\ r_n & \text{if } n \geq 0 \end{cases}$

$\therefore \begin{cases} x := r_1' \\ y := r_{-1}' \end{cases}$ satisfy $xy = w^N$ type A_{N-1} simple singularity

§3. torus action

$H_*^{G(\mathcal{O})}(\mathcal{R})$ has the homological degree



$\Longrightarrow \quad \mathbb{C}^\times \curvearrowright \mathcal{M}_C$ not necessarily conical

as it is not $\mathbb{Z}_{\geq 0}$-degree in general.

<u>Rem</u>. Poisson bracket degree -2

$\pi_0(\mathcal{R}) = \pi_0(Gr_G) = \pi_1(G)$ $\quad \therefore$ $H_*^{G(\mathcal{O})}(\mathcal{R})$ is $\pi_1(G)$-graded.

$\therefore \quad \pi_1(G)^\wedge \curvearrowright \mathcal{M}_C$

Pontryagin dual

e.g. $G = GL_n \Rightarrow \pi_1(G) = \mathbb{Z} \Rightarrow \pi_1(G)^\wedge = \mathbb{C}^\times$

<u>Rem</u>. Poisson bracket is preserved.

<u>Remark</u> (symplectic duality)

$\mathrm{Hom}_{grp}(\mathbb{C}^\times, \pi_1(G)^\wedge) \cong \mathrm{Hom}_{grp}(G, \mathbb{C}^\times)$

$\uparrow$ appears for GIT quotient

$M /\!/\!/_{\chi} G$

$\mathbb{C}^\times$-action on $\mathcal{M}_C$ $\longleftrightarrow$ stability condition on $\mathcal{M}_H$ (Higgs branch)

§4. Flavor symmetry

Suppose N is a representation of a larger group $\tilde{G}$ containing G as a normal subgroup.

$$G_F := \tilde{G}/G \quad \text{flavor symmetry}$$

$\tilde{G}(\mathcal{O}) \curvearrowright \mathrm{Gr}_G, \mathcal{T}, \mathcal{R}$ etc

$\Longrightarrow \operatorname{Spec} H_*^{\tilde{G}(\mathcal{O})}(\mathcal{R})$: deformation of $\mathcal{M}_C$ parametrized by $\operatorname{Spec} H^*_{G_F}(pt) = \mathfrak{t}_F/W_F$

Assume $G_F = T_F$ torus

Consider $\mathcal{R} = \mathcal{R}_{\tilde{G},N}$ for $(\tilde{G}, N)$.

$$\mathcal{M}_C(\tilde{G}, N) = \operatorname{Spec} H_*^{\tilde{G}(\mathcal{O})}(\mathcal{R}_{\tilde{G},N}) \curvearrowleft \pi_1(\tilde{G})^\wedge \longleftarrow \pi_1(T_F)^\wedge = T_F^\vee$$

<u>Prop.</u> $\mathcal{M}_C(G, N) \cong \mathcal{M}_C(\tilde{G}, N) /\!/\!/ T_F^\vee$ (symplectic reduction)

One can also consider GIT quotient

$$\mathcal{M}_C(\tilde{G}, N) /\!/\!/_{\rho} T_F^\vee = \mu_F^{-1}(0) /\!/_{\rho} T_F^\vee \quad \text{for } \rho \in \operatorname{Hom}(T_F^\vee, \mathbb{C}^\times) \cong \operatorname{Hom}(\mathbb{C}^\times, T_F)$$

<u>Rem</u> (symplectic duality, 2^{nd} instance)

deformation / GIT quotient for $\mathcal{M}_C$ $\longleftrightarrow$ $\mathbb{C}^\times$-action on $\mathcal{M}_H = M /\!/\!/ G \curvearrowleft \tilde{G}/G = T_F$

$\rho \in \mathrm{Hom}(\mathbb{C}^\times, T_F)$

<u>Example</u> : Goto–Bielawski–Dancer–toric hyperKähler manifold

$$1 \to T \to (\mathbb{C}^\times)^n \to T_F \to 1 \qquad \mathbb{C}^n \oplus (\mathbb{C}^n)^* /\!/\!/_{\xi}\, T \quad \text{for } \xi : T \to \mathbb{C}^\times$$

$$(\mathbb{C}^\times)^n \curvearrowright N = \mathbb{C}^n$$

$$\mathcal{M}_C(T, N) = \underbrace{\mathcal{M}_C((\mathbb{C}^\times)^n, N)}_{\substack{\| \\ \mathbb{C}^n \oplus (\mathbb{C}^n)^* \\ \text{by Example 2}}} /\!/\!/\, T_F^\vee$$

: toric hyperKähler manifold associated with

$$1 \to T_F^\vee \to (\mathbb{C}^\times)^n \longrightarrow T^\vee \to 1$$

§5. Birational description ("classical" Coulomb branch)

- $H_*^{G(\mathcal{O})}(\mathcal{R}) \cong H_*^{T(\mathcal{O})}(\mathcal{R})^W$
- $H_*^{T(\mathcal{O})}(\mathcal{R}) \otimes_{H_T^*(pt)} \operatorname{Frac} H_T^*(pt) \underset{\text{localization thm}}{\cong} H_*^{T(\mathcal{O})}(\mathcal{R}^T) \otimes_{H_T^*(pt)} \operatorname{Frac} H_T^*(pt)$

- $(Gr_G)^T = Gr_T$
- $N(\underset{K}{\mathcal{O}})^T = N^T(\underset{K}{\mathcal{O}})$

$\Rightarrow \mathcal{R}^T = \mathcal{T}^T = \mathcal{T}$ for (T, N^T)

$\therefore \operatorname{Spec} H_*^{T(\mathcal{O})}(\mathcal{R}^T) = \mathcal{M}_C(T, \underbrace{N^T}_{\text{trivial rep.}}) \underset{\text{Example 1}}{\cong} \mathfrak{t} \times T^\vee$

$\mathfrak{t} = \operatorname{Lie} T$

$T \subset G$ max. torus

$$\therefore \begin{tikzcd} \mathcal{M}_C \arrow[rr, dashed, "\text{birational}"] \arrow[dr, "\varpi"'] & & \mathfrak{t} \times T^\vee / W \arrow[dl, "1^{st}\text{ proj.}"] \\ & \mathfrak{t}/W \arrow[u, phantom, "\circlearrowright"] & \end{tikzcd}$$

"classical" Coulomb branch

$\therefore$ general fiber of ϖ $= T^\vee$